Kuwait University	**Math 101**	**Date:**	**April 9, 2009**
Dept. of Math. & Comp. Sci.	**First Exam**	**Duration:**	**90 minutes**

Calculators, mobile phones, pagers and all other mobile communication equipments are not allowed

Answer the following questions:

1. Use the definition of the limit to show that $\lim_{x \to -3} (2x - 1) = -7$ (3 pts.)

2. Evaluate the following limits, if they exist:

   (a) $\lim_{x \to 3} \frac{\sqrt{x+1} - 2}{x^2 - 9}$ (3 pts.)

   (b) $\lim_{x \to 1} (x-1)^{\frac{2}{3}} \cos\left(\frac{1}{x-1}\right)$ (3 pts.)

3. Find the vertical and horizontal asymptotes, if any, for the graph of

$$f(x) = \frac{|x-1|}{x^4 - x}.$$ (4 pts.)

4. Find the x-coordinates of the points at which the function f is discontinuous, where

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & \text{if } x > 1, \\[2ex] \dfrac{\sin(x-1)}{x^2 - 1}, & \text{if } x < 1. \end{cases}$$

   Classify the types of discontinuity of f as removable, jump, or infinite. (4 pts.)

5. (a) Use the Intermediate Value Theorem to show that $f(x) = x^4 + x^3 - 3x + 7$ has a horizontal tangent line. (3pts.)

   (b) Show that the graph of $f(x) = \frac{x^{\frac{2}{3}}}{x-1}$ has a vertical tangent line. (3 pts.)

6. (a) State the definition of the derivative of the function f at $x = a$.

   (b) Evaluate the following limit, if it exists: $\lim_{x \to 1} \frac{x^{\frac{2}{5}} - 1}{x - 1}$. (2 pts.)

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1. (3 points) Use the definition of the limit to show that $\lim_{x\to -3}(2x-1)=-7$.

2. a) (3 points)

$$\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x^2-9}=\lim_{x\to 3}\left(\frac{\sqrt{x+1}-2}{x^2-9}\right)\left(\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}\right)$$

$$=\lim_{x\to 3}\left(\frac{(x+1)-4}{(x^2-9)(\sqrt{x+1}+2)}\right)=\lim_{x\to 3}\left(\frac{1}{(x+3)(\sqrt{x+1}+2)}\right)=\frac{1}{24}.$$

b) (3 points)

$$\lim_{x\to 1}(x-1)^{\frac{2}{3}}\cos\left(\frac{1}{x-1}\right)=0.$$

Since $-1\le\cos\left(\frac{1}{x-1}\right)\le 1$ we have $-(x-1)^{\frac{2}{3}}\le(x-1)^{\frac{2}{3}}\cos\left(\frac{1}{x-1}\right)\le(x-1)^{\frac{2}{3}}$.

As $\lim_{x\to 1}-(x-1)^{\frac{2}{3}}=\lim_{x\to 1}(x-1)^{\frac{2}{3}}=0$, then by the Sandwich theorem

$$\lim_{x\to 1}(x-1)^{\frac{2}{3}}\cos\left(\frac{1}{x-1}\right)=0.$$

3. (4 points) $f(x)=\dfrac{|x-1|}{x^4-x}=\dfrac{|x-1|}{x(x-1)(x^2+x+1)}$.

$$\lim_{x\to 1^+}f(x)=\lim_{x\to 1^+}\frac{(x-1)}{x(x-1)(x^2+x+1)}=\lim_{x\to 1^+}\frac{1}{x(x^2+x+1)}=\frac{1}{3}.$$

$$\lim_{x\to 1^-}f(x)=\lim_{x\to 1^-}\frac{-(x-1)}{x(x-1)(x^2+x+1)}=\lim_{x\to 1^-}\frac{-1}{x(x^2+x+1)}=-\frac{1}{3}.$$

$$\lim_{x\to 0^\pm}f(x)=\lim_{x\to 0^\pm}\frac{-(x-1)}{x(x-1)(x^2+x+1)}=\lim_{x\to 0^\pm}\frac{-1}{x(x^2+x+1)}=\mp\infty$$

Vertical asymptote: $x=0$.

$$\lim_{x\to\pm\infty}f(x)=\lim_{x\to\pm\infty^+}\frac{|x-1|}{x(x-1)(x^2+x+1)}=0.$$ Horizontal asymptote: $y=0$.

4. (4 points) $f(x)$ is discontinuous at $x=-1,\ 1,\ 2$.

$\lim_{x\to -1^\pm}f(x)=\lim_{x\to -1^\pm}\dfrac{\sin(x-1)}{x^2-1}=\lim_{x\to -1^\pm}\dfrac{\sin(x-1)}{(x+1)(x-1)}=\pm\infty$. Infinite discont. at $x=-1$.

$$\lim_{x\to 1^-}\frac{\sin(x-1)}{x^2-1}=\lim_{x\to 1^-}\frac{\sin(x-1)}{(x+1)(x-1)}=\frac{1}{2}.$$

$\lim_{x\to 1^+}\dfrac{x^2-4}{x-2}=3$. Jump discontinuity at $x=1$.

$\lim_{x\to 2}\dfrac{x^2-4}{x-2}=\lim_{x\to 2}\dfrac{(x+2)(x-2)}{x-2}=4$. Removable discontinuity at $x=2$.

5. (a) (3 points)

$f'(x) = 4x^3 + 3x^2 - 3.$

$f'(0) = -3 < 0$

$f'(1) = 4 > 0.$

$f'(x)$ is continuous everywhere, it is continuous on [0, 1]. As $f'(0)f'(1) < 0$, then by IVT there exists at least one $c \in (0,1)$ such that $f'(c) = 0$.

b) (3 points)

$$f(x) = \frac{x^{\frac{2}{3}}}{x-1}.$$

$$f'(x) = \frac{\frac{2}{3}x^{-\frac{1}{3}}(x-1) - x^{\frac{2}{3}}}{(x-1)^2} = \frac{2(x-1)-3x}{3x^{\frac{1}{3}}(x-1)^2} = \frac{-x-2}{3x^{\frac{1}{3}}(x-1)^2}.$$

$\lim_{x \to 0} |f'(x)| = \infty$ and $f(x)$ is continuous at $x = 0$.

Thus, $f(x)$ has a vertical tangent at $x = 0$.

6. a) (1 point) $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$

b) (1 point)

$$f(x) = x^{\frac{2}{5}}.$$

$$f'(x) = \frac{2}{5}x^{-\frac{3}{5}}$$

$$f'(1) = \lim_{x \to 1} \frac{x^{\frac{2}{5}} - 1}{x - 1} = \frac{2}{5}.$$